

# Weak Convergence of Equity Derivatives Pricing with Default Risk \*

Gaoxiu Qiao <sup>†</sup> and Qiang Yao <sup>‡</sup>

**Abstract** This paper presents a discrete-time equity derivatives pricing model with default risk in a no-arbitrage framework. Using the equity-credit reduced form approach where default intensity mainly depends on the firm's equity value, we deduce the Arrow-Debreu state prices and the explicit pricing result in discrete time after embedding default risk in the pricing model. We prove that the discrete-time defaultable equity derivatives pricing has convergence stability, and it converges weakly to the continuous-time pricing results.

**Key words:** Default Risk; Hazard Process; Weak Convergence

**AMS subject classifications.** 60B10, 91G80

## 1 Introduction

Default risk is the risk that the agents cannot fulfill their obligations in the contracts. The reduced form approach has become a standard tool for modeling default risk. It considers the default to be an exogenously specified jump process, and derives the default probability as the instantaneous likelihood of default, see, for example, Jarrow and Turnbull [11], Duffie and Singleton [7], Lando [12]. The default time is usually defined as the first jump time of a Cox process with a given intensity (hazard rate). Hence, these models are frequently called *intensity models*.

Recently, an alternative model named *equity-credit market approach* has emerged. It assumes that the default intensity depends on the firm's equity value(stock prices) and allows the stock price to jump to zero at the time of default. It has both reduced form and structural features. Default risk is incorporated in this equity modeling approach by assuming that the stock price  $S_t$  at time  $t$  can jump to zero with an intensity, which is assumed to be a function of  $S_t$ . The models described above are all continuous-time models, they are widely used to model default risk.

However, continuous-time models are often too complicated to handle, it is necessary to deduce discrete-time models and show that the pricing processes converge to the continuous-time models. This is not a trivial job, since weak convergence, by its nature, is not tied to a single probability space. Some authors have presented different discrete-time models for derivatives pricing and have established some weak convergence results. See, for example, Cox, Ross and Rubinstein [5], He [9], Duffie and Protter [6], Nieuwenhuis and Vellekoop [13], etc.

In this paper, our aim is to present a discrete-time equity derivatives pricing model with default

---

<sup>\*</sup>This work is partially supported by NSFC grants (No.71073129, No.11426187, No.11126236 and No.11201150), China Postdoctoral Science Foundation (No. 2013M541526), JBK.120211, the Fundamental Research Funds for Central Universities (No.239201278210075), and the 111 Project(B14019).

<sup>†</sup>The School of Finance, Southwestern University of Finance and Economics, Chengdu, Sichuan Province, 611130, P. R. China. Email: qiaogaoxiu@163.com

<sup>‡</sup>Corresponding Author. School of Finance and Statistics, East China Normal University, Shanghai 200241, P. R. China. Email: qyao@sfs.ecnu.edu.cn

risk in a no-arbitrage framework, and prove that the pricing in discrete-time converges weakly to the continuous-time pricing results. In comparison, our method is different from Nieuwenhuis and Vellekoop [13]. Following the discrete framework of He [9] and equity-credit market approach presented in [2], we describe the discrete-time pricing model in a no-arbitrage framework. After embedding default risk, we deduce the Arrow-Debreu state prices and the explicit pricing result in discrete time. In order to prove the weak convergence of pricing processes, several auxiliary results are presented.

The paper is organized as follows: In section 2, we introduce the continuous-time model using equity-credit reduce-from approach; In section 3, we illustrate a discrete-time model of the equity derivatives pricing with default risk; In section 4, weak convergence of equity derivatives pricing with default risk from discrete-time to continuous-time pricing is proved; Finally, in section 5, we summarize the article and make concluding remarks.

## 2 The continuous-time model

We first recall the continuous-time defaultable contingent claims pricing model. Given a probability space  $(\Omega, \mathcal{F}, P)$ ,  $T$  is a strictly positive real number which represents the final date,  $(\omega_t)_{0 \leq t \leq T}$  is a Brownian motion. Let  $\mathcal{F}_t = \sigma(\omega_s, s \leq t)$  for  $t \geq 0$ . We suppose  $\mathcal{F}_t \subset \mathcal{F}$  for all  $t$ , and  $P$  is the real-world probability. Furthermore, we denote by “ $\Rightarrow$ ” weak convergence from now on.

A default event occurs at a random time  $\tau$ , where  $\tau$  is a non-negative random variable. The default process is defined as  $N_t \triangleq \mathbf{1}_{\{\tau \leq t\}}$ , and  $\mathcal{H}_t = \sigma(N_s, s \leq t)$ , the filtration  $\mathcal{H}$  is used to describe the information about default time, where  $\mathcal{H} = \bigcup_{0 \leq t \leq T} \mathcal{H}_t$ . At any time  $t$ , the agent's information on the securities prices and default time is  $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$  and the agent knows whether or not the default has appeared. Hence, the default time  $\tau$  is a  $\mathcal{G}$  stopping time where  $\mathcal{G} = \bigcup_{0 \leq t \leq T} \mathcal{G}_t$ . In fact,  $\mathcal{G}$  is the smallest filtration which contains  $\mathcal{F}$  and allows  $\tau$  to be a stopping time. Assume that the pre-default stock price  $S_t$  has the following dynamics

$$dS_t = (b(S_t) + \lambda(S_t, t)S_t)dt + \sigma(S_t, t)S_t d\omega_t, \quad S_0 > 0. \quad (2.1)$$

Here we assume that  $b(x)$  is continuous,  $\sigma(S, t)$  is a positively bounded and nonsingular Borel-measurable function. In particular we have that  $\sigma(S, t) \geq \sigma$  for some positive constant  $\sigma$ ,  $\lambda(S, t)$  is a nonnegative, bounded, continuous,  $\mathcal{F}$ -progressively measurable and integrable function. The functions  $b(S)$ ,  $\lambda(S, t)S$  and  $\sigma(S, t)S$  are Lipschitz continuous in  $S$ , uniformly in  $t$ .

The bond price  $B_t$  satisfies  $dB_t = B_t r(S_t)dt$  and  $B_0 = 1$ , where  $r(x)$  is a nonnegative continuous function, representing the riskyless interest rate. Suppose there exists a constant  $K > 0$  such that  $|x^2 r(x)| \leq K(1 + x^2)$ .

There exists a  $\mathcal{G}$  equivalent martingale measure  $Q^*$  which is defined as  $dQ^*|_{\mathcal{F}_t} = \xi_t dP|_{\mathcal{F}_t}$ , where  $\xi_t$  is the Radon-Nikodým density satisfying

$$d\xi_t = \xi_t \theta(S_t) d\omega_t, \quad \xi_0 = 1. \quad (2.2)$$

Here  $\theta(x) = -\sigma(x)^{-1}(b(x) - r(x)x)$ . Define  $W_t$  via  $dW_t = d\omega_t - \theta(S_t)dt$ , then  $W_t$  is a Brownian motion with respect to  $\mathcal{F}$ , and under the changed measure

$$dS_t = S_t[(r(S_t) + \lambda(S_t, t))dt + \sigma(S_t, t)dW_t], \quad S_0 > 0. \quad (2.3)$$

Define  $G_t \triangleq Q^*(\tau > t \mid \mathcal{F}_t)$ ,  $\Gamma_t \triangleq -\ln G_t$ . We call  $\Gamma_t$  the  $\mathcal{F}$  hazard process of  $\tau$ . For the detailed

properties, one can refer to Bielecki and Rutkowski [3].

Let  $g(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  be a square integrable and measurable function, the equity derivatives are defined to be securities that pay  $g(S_T)$  dollars on the final date. This formulation subsumes all of the usual examples, such as the European options, convertible bonds and so on. The prices of equity derivatives at time  $t$  are

$$V(S_t, t) = \mathbf{1}_{\{\tau > t\}} \mathbb{E}_{Q^*} \left[ \frac{B_t e^{\Gamma_t}}{B_T e^{\Gamma_T}} g(S_T) \mid \mathcal{F}_t \right]. \quad (2.4)$$

Poisson process with stochastic intensity is called *Cox process*. Given  $\lambda(S_u, u)$ , denote by  $\{\bar{C}_t\}$  the Poisson process with intensity  $C_t = \int_0^t \lambda(S_u, u) du$ . Then  $\{\bar{C}_t\}$  is a Cox process. Following the equity-credit market models, the canonical construction of default time  $\tau$  under the Cox process  $\{\bar{C}_t\}$  is defined as  $\tau = \inf\{t \geq 0 : C_t \geq \Theta\}$ , where  $\Theta \sim \text{Exp}(1)$  and is independent of  $\mathcal{F}$  under  $Q^*$ . Then

$$Q^*(\tau > t \mid \mathcal{F}_t) = Q^*(\Theta > C_t \mid \mathcal{F}_t) = e^{-C_t}.$$

It is easy to see that under this condition, the default time is the first jump time of the Cox process, the  $\mathcal{F}$  hazard process of  $\tau$  satisfies

$$\Gamma_t = -\ln Q^*(\tau > t \mid \mathcal{F}_t) = -\ln Q^*(\Theta > C_t \mid \mathcal{F}_t) = C_t.$$

Let  $\Delta$  denote the bankruptcy state when the firm defaults at time  $\tau$ . Then we can also write the dynamics for the stock price subject to bankruptcy  $S_t^\Delta$  as follows:

$$dS_t^\Delta = S_t^\Delta [r(S_t)dt + \sigma(S_t, t)dW_t - dM_t],$$

where  $M_t = N_t - \int_0^{t \wedge \tau} \lambda(S_u, u) du$ , and  $M_t$  is a martingale. Moreover, referred to Hypothesis (H) in Blanchet-Scalliet and Jeanblanc [4]: all  $\mathcal{F}$ -martingales are  $\mathcal{G}$ -martingales. It implies that the  $\mathcal{F}$ -Brownian motion  $W_t$  remains a Brownian motion under the extended probability measure  $Q^*$  and with respect to the enlarged filtration  $\mathcal{G}$  and is independent of  $M_t$ .

Then we have

$$V(S_t, t) = \mathbf{1}_{\{\tau > t\}} \mathbb{E}_{Q^*} \left[ e^{-\int_t^T (r_s + \lambda_s) ds} g(S_T) \mid \mathcal{F}_t \right].$$

Here we write  $\sigma_t = \sigma(S_t)$ ,  $r_t = r(S_t)$ ,  $\lambda_t = \lambda(S_t, t)$  for simplicity. We can obtain the following result.

**Lemma 2.1.** Let  $Y(S_t, t) = \mathbb{E}_{Q^*} \left[ e^{-\int_t^T (r_s + \lambda_s) ds} g(S_T) \mid \mathcal{F}_t \right]$ . Then it satisfies

$$\frac{\partial Y}{\partial t} + \frac{\sigma_t^2 S_t^2}{2} \frac{\partial^2 Y}{\partial S^2} + (r_t + \lambda_t) S_t \frac{\partial Y}{\partial S} - (r_t + \lambda_t) Y = 0. \quad (2.5)$$

Proof. Let  $\hat{Y}(S_t, t) = \mathbb{E}_{Q^*} \left[ \frac{G_T g(S_T)}{B_T} \mid \mathcal{F}_t \right]$ . It can be regarded as the discount price of contingent claim  $G_T g(S_T)$  at time  $t$ , then it is  $\mathcal{F}$  martingale. By *Itô's formula*,

$$\frac{\partial \hat{Y}}{\partial t} + \frac{\sigma_t^2 S_t^2}{2} \frac{\partial^2 \hat{Y}}{\partial S^2} + (r_t + \lambda_t) S_t \frac{\partial \hat{Y}}{\partial S} = 0.$$

Since  $\hat{Y}(S_t, t) = e^{-\int_0^t (r_s + \lambda_s) ds} Y(S_t, t)$ , (2.5) is proved after using *Itô's formula* again.  $\square$

### 3 Discrete-time model in the defaultable market

For simplicity, the time horizon is assumed to be  $[0, 1]$  and divided into  $n$  steps, the length of every step is  $1/n$ . For  $k = 1, 2, \dots, n$ , let  $\varepsilon^k$  be a random variable on the probability space  $\bar{\Omega} = \{\omega_1, \omega_2\}$ . For example, set  $\varepsilon^k(\omega_1) = 1$ ,  $\varepsilon^k(\omega_2) = -1$ , and  $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$ .

Let  $\bar{\Omega}_n = \overbrace{\bar{\Omega} \times \bar{\Omega} \times \dots \times \bar{\Omega}}^n = \{\omega_1, \omega_2\}^n$ ,  $P_n = \overbrace{P \times P \times \dots \times P}^n$ . Then  $P_n$  is the probability measures defined on  $\bar{\Omega}_n$ , representing the real-world probability,  $\mathcal{F}_n$  is the filtration generated by  $\varepsilon^k, k = 1, \dots, n$ ,  $\{\varepsilon^1, \varepsilon^2, \dots, \varepsilon^n\}$  is a sequence of independent and identically distributed random vectors defined on  $\{\bar{\Omega}_n, \mathcal{F}_n, P_n\}$ .

There are two financial assets in the market: stock and bond. Since the increment of Brownian motion can be approximated by a sequence of independent and identically distributed random variables, the pre-default stock prices and bond prices can be written as

$$\begin{aligned} S_{k+1}^n &= S_k^n + \frac{b(S_k^n) + \lambda(S_k^n, \frac{k}{n})S_k^n}{n} + \frac{\sigma(S_k^n, \frac{k}{n})S_k^n}{\sqrt{n}}\varepsilon^{k+1}, \quad S_0^n = S_0. \\ B_{k+1}^n &= B_k^n \left(1 + \frac{r(S_k^n)}{n}\right), \quad B_0^n = 1. \end{aligned}$$

Here  $S_k^n, B_k^n$  denote the stock prices and bond prices at time  $\frac{k}{n}$  respectively. Let  $\tilde{S}_t^n = S_{[nt]}^n$ ,  $\tilde{B}_t^n = B_{[nt]}^n$ . Then  $\tilde{S}_t^n, \tilde{B}_t^n$  are Markov processes and have jumps only at time  $\frac{k}{n}$ . Moreover  $\tilde{S}_t^n$  can be expressed as follows

$$\begin{aligned} \tilde{S}_t^n &= S_0 + \sum_{i=0}^{[nt]-1} \frac{b(S_i^n) + \lambda(S_i^n, \frac{i}{n})S_i^n}{n} + \sum_{i=0}^{[nt]-1} \frac{\sigma(S_i^n, \frac{i}{n})S_i^n}{\sqrt{n}}\varepsilon^{i+1} \\ &= S_0 + \int_0^{\frac{[nt]}{n}} (b(\tilde{S}_u^n, u) + \lambda(\tilde{S}_u^n, u)\tilde{S}_u^n)du + \sum_{i=0}^{[nt]-1} \frac{\sigma(S_i^n, \frac{i}{n})S_i^n}{\sqrt{n}}\varepsilon^{i+1}. \end{aligned}$$

Similarly,

$$\tilde{B}_t^n = B_0 + \sum_{i=0}^{[nt]-1} \frac{r(S_i^n)B_i^n}{n} = B_0 + \int_0^{\frac{[nt]}{n}} r(\tilde{S}_u^n)\tilde{B}_u^n du.$$

The above discrete framework is employed in He [9] where he assumes the stock prices satisfying  $\pi(\omega_1; S_k^n)S_{k+1}^n(\omega_1) + \pi(\omega_2; S_k^n)S_{k+1}^n(\omega_2) = S_k^n$ , where  $\pi(\omega_s; S_k^n)$  ( $s = 1, 2$ ) are considered as the Arrow-Debreu state prices and the discount stock prices are martingale. He [9] gives the result that there exists unique equivalent martingale measure  $Q_n$  in discrete-time defaultable market, and  $dQ_n = \xi_n^n dP_n$ , where  $\xi_k^n = 2^k \pi_k^n B_k^n$ ,  $\pi_k^n$  is defined as the product of Arrow-Debreu state prices from 0 to  $k$ , the default-free discrete-time market is complete. Let  $\tilde{\xi}_t^n = \xi_{[nt]}^n$ , then  $\tilde{\xi}_t^n = \xi_0 + \sum_{i=0}^{[nt]-1} \frac{\theta(S_i^n)\xi_i^n}{\sqrt{n}}\varepsilon^{i+1}$ .

Suppose default occurs at random time  $\tau_n$ , where  $\tau_n$  is a non-negative random variable. The default process is defined as  $N_k^n = 1_{\{\tau_n \leq \frac{k}{n}\}}$ ,  $\sigma$  filtration  $\mathcal{H}_k^n = \sigma(N_i^n, 0 \leq i \leq k)$  and  $\mathcal{H}^n$  is used to describe the information about default time. At time  $\frac{k}{n}$ , the agent's information on the prices and on the default time is  $\mathcal{G}_k^n = \mathcal{F}_k^n \vee \mathcal{H}_k^n$ . Hence, the default time  $\tau_n$  is a  $\mathcal{G}^n$  stopping time where  $\mathcal{G}^n = \{\mathcal{G}_k^n, 0 \leq k \leq n\}$ .

From Blanchet-Scalliet and Jeanblanc [4], if the defaultfree market is complete and arbitrage-free, the defaultable market is arbitrage-free, then there exists an equivalent martingale measure  $Q_n^*$  in  $\mathcal{G}^n$ -market.

**Definition 3.1.** For any  $0 \leq k \leq n$ , Let  $Q_n^*(\tau_n = 0) = 0$ ,  $Q_n^*(\tau_n > \frac{k}{n}) > 0$ . We write  $F_k^n = Q_n^*(\tau_n \leq \frac{k}{n} \mid \mathcal{F}_k^n)$ ,  $G_k^n = 1 - F_k^n = Q_n^*(\tau_n > \frac{k}{n} \mid \mathcal{F}_k^n)$ . Suppose  $F_k^n < 1$ , Let  $\Gamma_k^n \triangleq -\ln G_k^n = -\ln(1 - F_k^n)$  is called

the  $\mathcal{F}^n$  hazard process of  $\tau_n$  under  $Q_n^*$ .

Several properties in the continuous-time model still hold here, such as  $(F_k^n)$  is nonnegative bounded submartingale,  $\Gamma_k^n$  is increasing, and  $L_k^n = 1_{\{\tau_n > \frac{k}{n}\}} e^{\Gamma_k^n}$  is a martingale, the detailed properties can refer to [4].

Assume the Cox process is defined via intensity  $\tilde{C}_t \triangleq \int_0^t \lambda(\tilde{S}_u^n, u) du$ ,  $\Theta$  is the random variable with an exponential law of parameter 1, which is independent of  $\mathcal{F}^n$  under  $Q_n^*$ , then we can give the canonical construction of  $\tau_n$ ,  $\tau_n : \bar{\Omega}_n \rightarrow [0, T]$ ,  $\tau_n = \inf \left\{ \frac{[nt]}{n} \geq 0; \tilde{C}_{\frac{[nt]}{n}} \geq \Theta \right\}$ . Therefore,

$$Q_n^* \left( \tau_n > \frac{[nt]}{n} \mid \mathcal{F}_{[nt]}^n \right) = Q_n^* \left( \Theta > \tilde{C}_{\frac{[nt]}{n}} \mid \mathcal{F}_{[nt]}^n \right) = e^{-\tilde{C}_{\frac{[nt]}{n}}}.$$

The  $\mathcal{F}^n$  hazard process  $\tau_n$  satisfy

$$\Gamma_k^n = -\ln Q_n^* \left( \tau_n > \frac{k}{n} \mid \mathcal{F}_k^n \right) = -\ln Q_n^* \left( \Theta > \tilde{C}_{\frac{k}{n}} \mid \mathcal{F}_k^n \right) = \tilde{C}_{\frac{k}{n}}.$$

Moreover,

$$\Gamma_{k+1}^n = \Gamma_k^n + \int_{\frac{k}{n}}^{\frac{k+1}{n}} \lambda(\tilde{S}_u^n, u) du = \Gamma_k^n + \frac{\lambda(S_k^n, \frac{k}{n})}{n}.$$

Define  $\tilde{\Gamma}_t^n = \Gamma_{[nt]}^n$ , then  $\tilde{\Gamma}_t^n$  is a sequence of Markov process on probability space  $(\bar{\Omega}_n, \mathcal{F}^n, Q_n^*)$  with sample path in  $D_{\mathbb{R}}[0, 1]$  and

$$\tilde{\Gamma}_t^n = \sum_{i=0}^{[nt]-1} \frac{\lambda(S_i^n, \frac{i}{n})}{n} = \int_0^{\frac{[nt]}{n}} \lambda(\tilde{S}_u^n, u) du, \quad \tilde{\Gamma}_0^n = 0.$$

Although  $\tau_n$  and  $\tau$  are defined in different probability spaces, the canonical construction provides us a feasible way to prove the weak convergence of the default process which will be shown in the next section.

Define auxiliary discount process

$$\beta_k^n = B_k^n e^{\Gamma_k^n} \triangleq \prod_{i=0}^{k-1} \left( 1 + \frac{\tilde{r}_i}{n} \right),$$

then we have  $\frac{\beta_{k+1}^n}{\beta_k^n} = \frac{B_{k+1}^n e^{\Gamma_{k+1}^n}}{B_k^n e^{\Gamma_k^n}} = \left( 1 + \frac{r(S_k^n)}{n} \right) \exp \left( \frac{\lambda(S_k^n)}{n} \right)$ , where  $\tilde{r}_k = (r_k + \lambda_k) + \frac{\lambda_k^2 + 2r_k\lambda_k}{2n} + o\left(\frac{1}{n}\right)$ . Here we write  $r_k = r(S_k^n)$  and  $\lambda_k = \lambda(S_k^n)$  for simplicity. Suppose the equity prices and bond prices satisfy

$$\tilde{\pi}(\omega_1; S_k^n) S_{k+1}^n(\omega_1) + \tilde{\pi}(\omega_2; S_k^n) S_{k+1}^n(\omega_2) = S_k^n, \quad \tilde{\pi}(\omega_1; S_k^n) B_{k+1}^n(\omega_1) + \tilde{\pi}(\omega_2; S_k^n) B_{k+1}^n(\omega_2) = B_k^n.$$

Solve the above two equations, we get  $\tilde{\pi}(\omega_s; S_k^n) = \frac{1}{2} \left( 1 + \frac{\theta(S_k^n)}{\sqrt{n}} \varepsilon^{k+1} \right) \left( 1 + \frac{\tilde{r}_k}{n} \right)^{-1}$ . Set

$$\tilde{\pi}_k^n = \tilde{\pi}(\cdot; S_{k-1}^n) \tilde{\pi}(\cdot; S_{k-2}^n) \cdots \tilde{\pi}(\cdot; S_0^n), \quad k = 1, 2, \dots, n \quad \tilde{\pi}_0^n = 1.$$

Since  $\theta$  is bounded, for  $n$  large enough,  $\tilde{\pi}$  is non-negative. Then

$$\tilde{\pi}(\omega_1; S_k^n) + \tilde{\pi}(\omega_2; S_k^n) = \left( 1 + \frac{\tilde{r}_k}{n} \right)^{-1}, \quad \xi_k^n = 2^n \tilde{\pi}_k^n \beta_k^n = 2^n \pi_k^n B_k^n.$$

The price of defaultable contingent claims  $g(S_n^n)$  at time  $\frac{k}{n}$  is

$$V^n \left( S_k^n, \frac{k}{n} \right) = 1_{\{\tau_n > \frac{k}{n}\}} \mathbb{E}_{Q^n} \left[ \frac{B_k^n e^{\Gamma_k^n}}{B_n^n e^{\Gamma_n^n}} g(S_n^n) \mid \mathcal{F}_k^n \right]. \quad (3.1)$$

**Lemma 3.1.** Let  $Y_n \left( S_k^n, \frac{k}{n} \right) = \mathbb{E}_{Q^n} \left[ \frac{\beta_k^n}{\beta_n^n} g(S_n^n) \mid \mathcal{F}_k^n \right]$ . Then the following equation holds:

$$Y_n \left( S_k^n, \frac{k}{n} \right) = \tilde{\pi}(\omega_1; S_k^n) Y_n \left( S_{k+1}^n(\omega_1), \frac{k}{n} \right) + \tilde{\pi}(\omega_2; S_k^n) Y_n \left( S_{k+1}^n(\omega_2), \frac{k}{n} \right). \quad (3.2)$$

Proof. Since  $Y_n \left( S_k^n, \frac{k}{n} \right)$  is a  $\mathcal{F}^n$  martingale, then we have

$$\begin{aligned} Y_n \left( S_k^n, \frac{k}{n} \right) &= \mathbb{E}_{Q^n} \left[ Y_n \left( S_{k+1}^n, \frac{k+1}{n} \right) \left( 1 + \frac{\tilde{r}_k}{n} \right)^{-1} \mid \mathcal{F}_k^n \right] \\ &= \mathbb{E}_{P_n} \left[ \frac{\xi_{k+1}^n}{\xi_k^n} Y_n \left( S_{k+1}^n, \frac{k+1}{n} \right) \left( 1 + \frac{\tilde{r}_k}{n} \right)^{-1} \mid \mathcal{F}_k^n \right] = \mathbb{E}_{P_n} \left[ \frac{2\pi_{k+1}^n B_{k+1}^n}{\pi_k^n B_k^n} Y_n \left( S_{k+1}^n, \frac{k+1}{n} \right) \left( 1 + \frac{\tilde{r}_k}{n} \right)^{-1} \mid \mathcal{F}_k^n \right] \\ &= \frac{1}{2} \left( 1 + \frac{\theta(S_k^n)}{\sqrt{n}} \right) \left( 1 + \frac{\tilde{r}_k}{n} \right)^{-1} Y_n \left( S_{k+1}^n(\omega_1), \frac{k+1}{n} \right) + \frac{1}{2} \left( 1 - \frac{\theta(S_k^n)}{\sqrt{n}} \right) \left( 1 + \frac{\tilde{r}_k}{n} \right)^{-1} Y_n \left( S_{k+1}^n(\omega_2), \frac{k+1}{n} \right) \\ &= \tilde{\pi}(\omega_1; S_k^n) Y_n \left( S_{k+1}^n(\omega_1), \frac{k+1}{n} \right) + \tilde{\pi}(\omega_2; S_k^n) Y_n \left( S_{k+1}^n(\omega_2), \frac{k+1}{n} \right). \end{aligned}$$

Therefore, (3.2) is proved.  $\square$

We can conclude that  $\tilde{\pi}(\cdot; S_k^n)$  can be regarded as the discrete-time Arrow-Debreu state prices in the defaultable market.

## 4 The main result: Weak Convergence

In this section, we will prove the weak convergence of pricing process for defaultable equity derivatives under the above model. Firstly, we introduce infinite dimensional multiplicative probability space  $\bar{\Omega}_N \triangleq \overbrace{\bar{\Omega} \times \bar{\Omega} \times \cdots \times \bar{\Omega}}^{\infty}$ , then  $\bar{\Omega}_n$  is a subspace of  $\bar{\Omega}_N$ . From the infinite multiply probability existence theorem, there exists a unique probability measure  $\hat{P}$  satisfying condition:  $\hat{P}(A \times \bar{\Omega}_{N \setminus I_n}) = Q_n(A)$ , where  $A \in \bar{\Omega}_n$ ,  $I_n = \{1, 2, \dots, n\}$ .

He [9] proves the weak convergence of Markov process vector including equity prices, bond prices, Radon-Nikodým density. Now we extend this result in the defaultable market. Combining Martingale central limit theorem developed by Ethier and Kurtz [8] (Page 354), we get a similar result.

**Lemma 4.1.** For any  $t \in [0, 1]$ , let  $\tilde{Z}_t^n = (\tilde{S}_t^n, \tilde{B}_t^n, \tilde{\xi}_t^n, \tilde{\Gamma}_t^n)$ ,  $Z_t = (S_t, B_t, \xi_t, \Gamma_t)$ . Then

$$\tilde{Z}_t^n \Rightarrow Z_t \text{ as } n \rightarrow \infty.$$

Recall the definition of  $\tau_n$  and  $\tau$ , the following conclusion holds.

**Lemma 4.2.** For  $t \in [0, 1]$ , let  $X_n(t) = 1_{\{\tau_n > \frac{\lfloor nt \rfloor}{n}\}}$ ,  $X(t) = 1_{\{\tau > t\}}$ . Then  $X_n(\cdot) \Rightarrow X(\cdot)$ .

Proof. For any  $t \in [0, 1]$ , we have  $\sup_n \mathbb{E}_{\hat{P}} [|X_n(t)|] = \sup_n \mathbb{E}_{Q^n} [|X_n(t)|] \leq 1 < \infty$ . Therefore,

$$\lim_{C \rightarrow \infty} \limsup_n \hat{P}(|X_n(t)| > C) \leq \lim_{C \rightarrow \infty} \frac{\limsup_n \mathbb{E}_{\hat{P}} [|X_n(t)|]}{C} = 0.$$

So  $\{X_n(t)\}$  is tight for any  $t \in [0, 1]$ .

Choose sequences  $\{\alpha_n\}$  and  $\{\delta_n\}$  satisfying the following: for all  $n$ ,  $\alpha_n$  is a stopping time with respect to the  $\sigma$  filtration which is generated by the process  $\{X_n(t) : 0 \leq t \leq 1\}$ , and  $\alpha_n$  has only finite value;  $\delta_n$  is a constant and  $0 \leq \delta_n \leq 1$ . Moreover  $\delta_n \rightarrow 0$ , as  $n \rightarrow \infty$ .

$$\begin{aligned} \hat{P}(|X_n(\alpha_n + \delta_n) - X_n(\alpha_n)| > \epsilon) &= Q_n(|X_n(\alpha_n + \delta_n) - X_n(\alpha_n)| > \epsilon) \\ &\leq \frac{1}{\epsilon} \mathbb{E}_{Q_n} [|X_n(\alpha_n + \delta_n) - X_n(\alpha_n)|] \leq \frac{1}{\epsilon} \mathbb{E}_{Q_n} \left[ 1_{\{\tau_n > \frac{[n\alpha_n]}{n}\}} - 1_{\{\tau_n > \frac{[n\alpha_n + n\delta_n]}{n}\}} \right] \\ &= \frac{1}{\epsilon} E_{Q_n} \left[ \mathbb{E}_{Q_n} \left[ 1_{\{\tau_n > \frac{[n\alpha_n]}{n}\}} \mid \mathcal{F}_{[n\alpha_n]}^n \right] - \mathbb{E}_{Q_n} \left[ 1_{\{\tau_n > \frac{[n\alpha_n + n\delta_n]}{n}\}} \mid \mathcal{F}_{[n\alpha_n + n\delta_n]}^n \right] \right] \\ &= \frac{1}{\epsilon} \mathbb{E}_{Q_n} \left[ \exp(-\Gamma_{[n\alpha_n]}^n) - \exp(-\Gamma_{[n\alpha_n + n\delta_n]}^n) \right] \\ &= \frac{1}{\epsilon} \mathbb{E}_{Q_n} \left[ \exp \left( - \sum_{i=0}^{[n\alpha_n]-1} \frac{\lambda(S_i^n)}{n} \right) \left( 1 - \exp \left( - \sum_{i=[n\alpha_n]}^{[n\alpha_n + n\delta_n]-1} \frac{\lambda(S_i^n)}{n} \right) \right) \right]. \end{aligned}$$

Since  $\lambda(S, t)$  is a nonnegative *bounded* continuous function, we have  $\sum_{i=[n\alpha_n]}^{[n\alpha_n + n\delta_n]-1} \frac{\lambda(S_i^n)}{n} \leq C\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ , where  $C$  is a fixed constant. Together with the fact that  $\exp \left( - \sum_{i=0}^{[n\alpha_n]-1} \frac{\lambda(S_i^n)}{n} \right) \leq 1$ , we have

$\hat{P}(|X_n(\alpha_n + \delta_n) - X_n(\alpha_n)| > \epsilon) \rightarrow 0$ , that is,  $X_n(\alpha_n + \delta_n) - X_n(\alpha_n) \xrightarrow{\hat{P}} 0$  as  $n \rightarrow \infty$ . By the criterion of Aldous [1] (Page 1),  $\{X_n(\cdot)\}$  is tight in  $D_{\mathbb{R}}[0, 1]$ .

We have  $X_n(t) = \mathbf{1}_{\{\tau_n > \frac{[nt]}{n}\}} = \mathbf{1}_{\{\Theta > \tilde{\Gamma}_{\frac{[nt]}{n}}\}} = \mathbf{1}_{\{\Theta > \tilde{\Gamma}_t\}}$  and  $X(t) = \mathbf{1}_{\{\tau > t\}} = \mathbf{1}_{\{\Theta > C_t\}} = \mathbf{1}_{\{\Theta > \Gamma_t\}}$  for any  $t \in [0, 1]$ . Since  $\tilde{\Gamma}^n \Rightarrow \Gamma$  as  $n$  tends to infinity, we have  $\mathbb{E}_{Q_n}[e^{iuX_n(t)}] \rightarrow \mathbb{E}_Q[e^{iuX(t)}]$  as  $n$  tends to infinity. According to the dominated convergence theorem, for any  $t_1, t_2, \dots, t_m \in [0, 1]$ ,  $u_1, \dots, u_m \in \mathbb{R}$ ,

$$\mathbb{E}_{Q_n} \left[ e^{i \sum_{j=1}^m u_j X_n(t_j)} \right] \rightarrow \mathbb{E}_Q \left[ e^{i \sum_{j=1}^m u_j X(t_j)} \right], \quad n \rightarrow \infty.$$

Therefore,  $\{X_n\}$  is tight, and their finite dimensional distribution converges. From Ethier and Kurtz [8] (Page 131),  $\{X_n\}$  converges weakly to  $X$ .  $\square$

**Lemma 4.3.** For any integers  $l, m, k \geq 0$ ,  $l \leq k \leq n$ , there exists a constant  $C > 0$ , depending on  $m$  (large enough), such that  $\mathbb{E}_{Q_n} [S_k^n]^{2m} \mid \mathcal{F}_l^n \leq C(1 + [S_0]^{2m})$ .

Next we prove the weak convergence of the second part in the equation of defaultable contingent claims prices, following a similar argument to the main theorem of He [9].

**Lemma 4.4.** For any  $t \in [0, 1]$ , let

$$Y(t) \triangleq Y(S_t, t) = \mathbb{E}_Q \left[ \frac{B_t e^{\Gamma_t}}{B_T e^{\Gamma_T}} g(S_T) \mid \mathcal{F}_t \right], \quad Y_n(t) \triangleq Y_n \left( S_{[nt]}^n, \frac{[nt]}{n} \right) = \mathbb{E}_{Q_n} \left[ \frac{B_{[nt]}^n e^{\Gamma_{[nt]}^n}}{B_n^n e^{\Gamma_n^n}} g(S_n^n) \mid \mathcal{F}_{[nt]}^n \right].$$

Suppose that  $Y$  is continuously differentiable up to the third order and that  $Y$  and all of its derivatives up to the third order satisfy a polynomial growth condition. Then  $Y_n(\cdot) \Rightarrow Y(\cdot)$  as  $n$  tends to infinity.

**Remark.** We get the idea of the proofs of Lemma 4.3 and Lemma 4.4 from He [9], but the results in our paper are rather different from them. In Lemma 4.3, we give the inequality for a more general case. In Lemma 4.4, we prove that  $\tilde{e}_t^n$  converges to zero in the sense of almost everywhere.

Then combine Lemma 4.2 and Lemma 4.4, we obtain the main result.

**Theorem 4.1.** Suppose  $g(\cdot)$  is  $\mathbb{R} \rightarrow \mathbb{R}$  square integrable measurable function. Let  $\tilde{V}^n(\tilde{S}_t^n, t) = V^n(S_{[nt]}, \frac{[nt]}{n})$ ,  $V(S_t, t)$  and  $\tilde{V}^n(\tilde{S}_t^n, \frac{[nt]}{n})$  satisfy (2.4) and (3.1) respectively. Then we have

$$\tilde{V}^n(\tilde{S}_t^n, \cdot) \Rightarrow V(S, \cdot), \quad \text{as } n \rightarrow \infty.$$

Proof. Clearly, we have  $\tilde{V}^n(\tilde{S}_t^n, \frac{[nt]}{n}) = X_n(t)Y_n(t)$  and  $V(S_t, t) = X(t)Y(t)$ . By Lemma 4.4,  $Y_n$  converges weakly to  $Y$ , then  $Y_n$  is relatively tight. Since  $\mathbb{R}$  is separable and  $(\mathbb{R}, d)$  is complete, then  $D_{\mathbb{R}}[0, 1]$  is separable, it follows that  $\{Y_n\}$  is tight.

By Lemma 4.2,  $\{X_n\}$  is tight, together with the fact that  $Y(t) = Y(S_t, t)$  is continuous with respect to  $t$ , then  $\{(X_n, Y_n)\}$  is tight according to Jacod-Shiryaev [10] (Page 353).

Next, we only need to prove the convergence of their finite dimension distribution. That is, for any  $u_1, \dots, u_m \in \mathbb{R}$ ,  $v_1, \dots, v_m \in \mathbb{R}$ ,

$$\mathbb{E}_{Q_n} \left[ e^{i \sum_{j=1}^m (u_j X_n(t_j) + v_j Y_n(t_j))} \right] \rightarrow \mathbb{E}_Q \left[ e^{i \sum_{j=1}^m (u_j X(t_j) + v_j Y(t_j))} \right], \quad n \rightarrow \infty. \quad (4.1)$$

From Lemma 4.2 and Lemma 4.4, we can obtain the convergence of the finite dimension distribution of  $\{X_n\}$ ,  $\{Y_n\}$ . Moreover,  $X_n(t)$  and  $Y_n(t)$  are measurable with respect to  $\mathcal{H}_t$  and  $\mathcal{F}_t$  respectively, and  $X_n(t)$ ,  $Y_n(t)$  are independent, (4.1) holds clearly. Then  $(X_n, Y_n) \Rightarrow (X, Y)$  as  $n \rightarrow \infty$ .

Let  $f(X_n(t)Y_n(t)) = X_n(t)Y_n(t)$ , where  $f$  is a continuous function that maps  $X_n(t)$ ,  $Y_n(t)$  from  $D_{\mathbb{R}}[0, 1] \times D_{\mathbb{R}}[0, 1]$  to  $D_{\mathbb{R}}[0, 1]$ . By continuous mapping theorem (Ethier-Kurtz [8], p.354),  $X_n(\cdot)Y_n(\cdot) \Rightarrow X(\cdot)Y(\cdot)$  as  $n \rightarrow \infty$ .  $\square$

In the following, we give the details of proofs of Lemma 4.1, 4.3 and 4.4. For simplicity, we write  $b(\tilde{S}_u^n) = b_u$ ,  $\lambda(\tilde{S}_u^n, u) = \lambda_u$ ,  $\sigma(\tilde{S}_u^n) = \sigma_u$ ,  $\theta(\tilde{S}_u^n) = \theta_u$  in the proof.

**Proof of Lemma 4.1.**  $d\Gamma_t = \lambda(S_t, t)dt$ . Since  $b(\cdot)$ ,  $\lambda(\cdot)S$ ,  $\sigma(\cdot)S$  satisfy the Lipschitz condition, therefore (2.1) has a unique solution, which implies that (2.2), (2.3) also have unique solutions respectively. Since  $\{\tilde{S}^n\}$ ,  $\{\tilde{B}^n\}$ ,  $\{\tilde{\xi}^n\}$  and  $\{\tilde{\Gamma}^n\}$  are processes that are right continuous with left limits, hence  $\{Z^n\}$  is a sequence of Markov process vectors with sample path in  $D_{\mathbb{R}^4}[0, 1]$ , where  $D_{\mathbb{R}^4}[0, 1]$  is the space of functions from  $[0, 1]$  to  $\mathbb{R}^4$ , right continuous with left limits. Denote  $L_t^n$  and  $A_t^n$  by

$$L_t^n = \begin{pmatrix} \int_0^{\frac{[nt]}{n}} (b_u + \lambda_u \tilde{S}_u^n) du \\ \int_0^{\frac{[nt]}{n}} r \tilde{B}_u^n du \\ 0 \\ \int_0^{\frac{[nt]}{n}} \lambda_u du \end{pmatrix}, A_t^n = \begin{pmatrix} \int_0^{\frac{[nt]}{n}} \sigma_u^2 S^2 du & 0 & \int_0^{\frac{[nt]}{n}} \sigma_u S \theta_u \tilde{\xi}_u^n du & 0 \\ 0 & 0 & 0 & 0 \\ \int_0^{\frac{[nt]}{n}} \sigma_u S \theta_u \tilde{\xi}_u^n du & 0 & \int_0^{\frac{[nt]}{n}} (\theta_u \tilde{\xi}_u^n)^2 du & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then  $\{L^n\}$ ,  $\{A^n\}$  are  $4 \times 1$  and  $4 \times 4$  (symmetric) matrix valued process respectively, and each of their elements has a sample path in  $D_{\mathbb{R}}[0, 1]$ . Moreover,  $A_t^n - A_s^n$  is non-negative definite for  $t > s \geq 0$ . Define

$$\tau_n^q := \inf\{t \leq T : |Z_t^n| \geq q \text{ or } |Z_{t-}^n| \geq q\}.$$

Next, we prove the four conditions for martingale central limit theorem holds.

(a) It is directly from  $Z_0^n = Z_0 = (S_0, B_0, \xi_0, 0)$ .

$$(b) M_t^n = \tilde{Z}^n - L_t^n = \begin{pmatrix} S_0 + \sum_{i=0}^{[nt]-1} \frac{\sigma(S_i^n) S_i^n}{\sqrt{n}} \varepsilon^{i+1} \\ B_0 \\ \xi_0 + \sum_{i=0}^{[nt]-1} \frac{\theta(S_i^n) \xi_i^n}{\sqrt{n}} \varepsilon^{i+1} \\ 0 \end{pmatrix}.$$

Let  $N_k = S_0 + \sum_{i=0}^{k-1} \frac{\sigma(S_i^n)S_i^n}{\sqrt{n}} \varepsilon^{i+1}$ . Clearly,

$$\mathbb{E}_{Q_n}[N_{k+1} - N_k \mid S_k^n] = \mathbb{E}_{Q_n}\left[\frac{\sigma(S_k^n)S_k^n}{\sqrt{n}} \varepsilon^{k+1} \mid S_k^n\right] = \frac{\sigma(S_k^n)S_k^n}{\sqrt{n}} \mathbb{E}_{Q_n}[\varepsilon^{k+1} \mid S_k^n] = 0.$$

So  $N_k$  are martingales. By the same arguments,  $M_t^n$  and  $M_t^n(M_t^n)^T - A_t^n$  are also martingales.

$$(c) \tilde{Z}_t^n - \tilde{Z}_{t-}^n = \begin{cases} \begin{pmatrix} \frac{b(S_{k-1}^n) + \lambda(S_{k-1}^n)S_{k-1}^n}{n} + \frac{\sigma(S_{k-1}^n)S_{k-1}^n}{\sqrt{n}} \varepsilon^k \\ \frac{r(S_{k-1}^n)B_{k-1}^n}{n} \\ \frac{\theta(S_{k-1}^n)\xi_{k-1}^n}{\lambda(S_{k-1}^n)} \varepsilon^k \\ \frac{1}{n} \\ 0 \end{pmatrix} & t = \frac{k}{n}, \\ \frac{k}{n} < t < \frac{k+1}{n}. \end{cases}$$

By the definition of  $\tau_n^q$ , when  $t \leq \tau_n^q$ ,  $|Z_t^n - Z_{t-}^n| \leq 2q$ , and  $|Z_t^n - Z_{t-}^n|^2$  is of order  $\frac{1}{n}$ . By the dominated convergence theorem, we have  $\lim_{n \rightarrow \infty} \mathbb{E}_n \left[ \sup_{t \leq \tau_n^q} |Z_t^n - Z_{t-}^n|^2 \right] = 0$ . Then using the same argument, we can

$$\text{get } \lim_{n \rightarrow \infty} \mathbb{E}_n \left[ \sup_{t \leq \tau_n^q} |L_t^n - L_{t-}^n|^2 \right] = 0 \text{ and } \lim_{n \rightarrow \infty} \mathbb{E}_n \left[ \sup_{t \leq \tau_n^q} |A_t^n - A_{t-}^n| \right] = 0.$$

(d) For all  $q > 0$ , we have

$$L_t^n - \begin{pmatrix} \int_0^t (b(\tilde{S}_u^n) + \lambda(\tilde{S}_u^n)\tilde{S}_u^n) du \\ \int_0^t r(\tilde{S}_u^n)\tilde{B}_u^n du \\ 0 \\ \int_0^t \lambda(\tilde{S}_u^n) du \end{pmatrix} = - \begin{pmatrix} (b(\tilde{S}_t^n) + \lambda(\tilde{S}_t^n)\tilde{S}_t^n) \left(t - \frac{[nt]}{n}\right) \\ r(\tilde{S}_t^n)\tilde{B}_t^n \left(t - \frac{[nt]}{n}\right) \\ 0 \\ \lambda(\tilde{S}_t^n) \left(t - \frac{[nt]}{n}\right) \end{pmatrix},$$

where  $\frac{k}{n} < t < \frac{k+1}{n}$ , and the above equation equals to zero when  $t = \frac{k}{n}$ . Therefore,

$$Q_n \left[ \sup_{t \leq \tau_n^q} \left| L_t^n - \int_0^t b(X_s^n) ds \right| \geq \epsilon \right] \leq \frac{\mathbb{E}_{Q_n} \left[ \sup_{t \leq \tau_n^q} \left| L_t^n - \int_0^t b(X_s^n) ds \right| \right]}{\epsilon}.$$

We can easily prove that  $\lim_{n \rightarrow \infty} \hat{P} \left[ \sup_{t \leq \tau_n^q} \left| L_t^n - \int_0^t b(X_s^n) ds \right| \geq \epsilon \right] = 0$  as  $n$  tends to infinity, as desired.  $\square$

**Proof of Lemma 4.3.** Applying Taylor's expansion to the function  $x^{2m}$ , we obtain

$$\begin{aligned} & [S_{k+1}^n]^{2m} \\ &= [S_k^n]^{2m} + 2m[S_k^n]^{2m-1}(S_{k+1}^n - S_k^n) + m(2m-1)[\bar{S}_k^n]^{2m-2}(S_{k+1}^n - S_k^n)^2 \\ &= [S_k^n]^{2m} + 2m[S_k^n]^{2m-1} \left( \frac{b + \lambda S}{n} + \frac{\sigma S}{\sqrt{n}} \varepsilon^{k+1} \right) + m(2m-1)[\bar{S}_k^n]^{2m-2} \left( \frac{b + \lambda S}{n} + \frac{\sigma S}{\sqrt{n}} \varepsilon^{k+1} \right)^2, \end{aligned}$$

where

$$\bar{S}_k^n = S_k^n + \beta \left( \frac{b(S_k^n)}{n} + \frac{\sigma(S_k^n)}{\sqrt{n}} \varepsilon^{k+1} \right), \quad \beta \in [0, 1].$$

Moreover,

$$\begin{aligned}
\mathbb{E}_{Q_n}[\varepsilon^{k+1} \mid \mathcal{F}_k^n] &= \mathbb{E}_{P_n} \left[ \frac{2\tilde{\pi}_{k+1}^n \beta_{k+1}^n}{\tilde{\pi}_k^n \beta_k^n} \varepsilon^{k+1} \mid \mathcal{F}_k^n \right] \\
&= \mathbb{E}_{P_n} \left[ \left( 1 + \frac{\theta(S_k^n)}{\sqrt{n}} \varepsilon^{k+1} \right) \varepsilon^{k+1} \mid \mathcal{F}_k^n \right] \\
&= \frac{1}{2} \left( 1 + \frac{\theta(S_k^n)}{\sqrt{n}} \right) - \frac{1}{2} \left( 1 - \frac{\theta(S_k^n)}{\sqrt{n}} \right) = \frac{\theta(S_k^n)}{\sqrt{n}}.
\end{aligned}$$

Then

$$\frac{b + \lambda S}{n} + \frac{\sigma S}{\sqrt{n}} \mathbb{E}_{P_n} [\varepsilon^{k+1} \mid \mathcal{F}_k^n] = \frac{b + \lambda S}{n} + \frac{\sigma S \theta}{n} = \frac{(r + \lambda)S}{n}.$$

Notice that

$$|\bar{S}_k^n| \leq |S_k^n| + |b| + |\lambda S| + |\sigma S|, \quad |\varepsilon^{k+1}| = 1,$$

and

$$x^{2m-2} \leq 1 + x^{2m}, \quad (x + y)^m \leq 2^m(x^m + y^m), \quad x^2 r(x) \leq K(1 + x^2)$$

when  $x, y > 0$ . Taking the conditional expectation with respect to  $\mathcal{F}_k^n$  under  $Q_n$ , we have

$$\begin{aligned}
\mathbb{E}_{Q_n} [[S_{k+1}^n]^{2m} \mid \mathcal{F}_k^n] &\leq [S_k^n]^{2m} + 2m[S_k^n]^{2m-1} \left( \frac{|r + \lambda||S|}{n} \right) \\
&\quad + \frac{m(2m-1)}{n} (|S_k^n| + |b| + |\lambda S| + |\sigma S|)^{2m-2} (|b| + |\lambda S| + |\sigma S|)^2.
\end{aligned}$$

Given the conditions on  $b, \lambda S, \sigma S$ , we can find a constant  $K' > 0$ , such that for any  $x \in R$ ,

$$|b(x)| \leq K'(1 + |x|), \quad |\lambda(x)x| \leq K'(1 + |x|), \quad |\sigma(x)x| \leq K'(1 + |x|), \quad |b(x)|^2 \leq K'(1 + x^2),$$

$$|\lambda(x)x|^2 \leq K'(1 + x^2), \quad |\sigma(x)x|^2 \leq K'(1 + x^2), \quad \text{and} \quad |x^2 r(x)| \leq K'(1 + x^2).$$

Hence we can obtain

$$\begin{aligned}
&\mathbb{E}_{Q_n} [[S_{k+1}^n]^{2m} \mid \mathcal{F}_k^n] \\
&\leq [S_k^n]^{2m} + \frac{2mK'}{n} (1 + 2[S_k^n]^{2m}) + \frac{9K'^2 m(2m-1)}{n} (3K' + (1 + 3K')[S_k^n])^{2m-2} (1 + [S_k^n]^2) \\
&\leq [S_k^n]^{2m} + \frac{2mK'}{n} (1 + 2[S_k^n]^{2m}) + \frac{9(2 + 6K')^{2m-2} m(2m-1)}{n} (1 + [S_k^n]^{2m-2}) (1 + [S_k^n]^2) \\
&\leq K/n + (1 + K/n)(1 + [S_k^n]^{2m}),
\end{aligned}$$

and furthermore,  $\mathbb{E}_{Q_n} [[S_k^n]^{2m} \mid \mathcal{F}_l^n] \leq (1 + K/n)^{k-l} (1 + [S_l^n]^{2m}) \leq A(1 + [S_l^n]^{2m})$ , where  $K$  depends on  $K'$  and  $m$ ,  $0 \leq l \leq k$ , and  $A = \sup_n (1 + K/n)^{k-l}$ .

Since  $\mathbb{E}_{Q_n} [S_{k+1}^n \mid \mathcal{F}_k^n] \geq \mathbb{E}_{Q_n} \left[ S_{k+1}^n \frac{B_k^n}{B_{k+1}^n} \mid \mathcal{F}_k^n \right] = S_k^n$ , we can get that  $(S_k^n)$  is a submartingale.

Moreover,  $\varphi(x) = x^{2m}$  is a convex and increasing function in  $\mathbb{R}^+$  and  $(S_k^n)$  is nonnegative. By Jensen's inequality, we have  $\mathbb{E}_{Q_n} [(S_{k+1}^n)^{2m} \mid \mathcal{F}_k^n] \geq \mathbb{E}_{Q_n} [S_{k+1}^n \mid \mathcal{F}_k^n]^{2m} \geq (S_k^n)^{2m}$ . It is easy to see that  $((S_k^n)^{2m})$

is a submartingale. By submartingale inequality, we have

$$\begin{aligned}\mathbb{E}_{Q_n}[[S_k^n]^{2m} \mid \mathcal{F}_l^n] &\leq A(1 + \mathbb{E}_{Q_n}[\sup_{0 \leq t \leq T} |\tilde{S}_t^n|^{2m}]) \leq A(1 + \left(\frac{2m}{2m-1}\right)^{2m} \mathbb{E}_{Q_n}[(\tilde{S}_T^n)^{2m}]) \\ &\leq A(1 + \left(\frac{2m}{2m-1}\right)^{2m} A(1 + (S_0^n)^{2m})) \leq C(1 + (S_0)^{2m}),\end{aligned}$$

where  $C$  is large enough, and  $\tilde{S}_T^n = S_n^n$ .  $\square$

**Proof of Lemma 4.4.** By Lemma 4.1,  $\tilde{Z}^n$  converges weakly to  $Z$ ,  $Y$  is a continuous function of  $\tilde{Z}^n$ . Applying continuous mapping theorem, we get

$$Y\left(\tilde{S}_\cdot^n, \frac{[n]}{n}\right) \Rightarrow Y(S_\cdot, \cdot), \quad n \rightarrow \infty. \quad (4.2)$$

Since  $Y_n\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) = Y\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - \tilde{e}_t^n$ , where  $\tilde{e}_t^n = Y\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - Y_n\left(\tilde{S}_t^n, \frac{[nt]}{n}\right)$ , we need only prove that the stochastic process  $\tilde{e}_\cdot^n$  converges weakly to zero.

Let “+” and “-” denote the states  $\varepsilon^{k+1} = 1$  and  $\varepsilon^{k+1} = -1$  respectively, and define  $S_{k+1}^{n+} = S_{k+1}^n(\omega_1)$ ,  $S_{k+1}^{n-} = S_{k+1}^n(\omega_2)$ . We define two functions as follows.

$$f_+^{k,n}(t) = Y(S_k^n + t(S_{k+1}^{n+} - S_k^n), t_k^n + t(t_{k+1}^n - t_k^n)), \quad f_-^{k,n}(t) = Y(S_k^n + t(S_{k+1}^{n-} - S_k^n), t_k^n + t(t_{k+1}^n - t_k^n)),$$

Let

$$\frac{\partial Y}{\partial S} = Y_S, \quad \frac{\partial Y}{\partial t} = Y_t, \quad \frac{\partial^2 Y}{\partial S^2} = Y_{SS}, \quad \frac{\partial^2 Y}{\partial t^2} = Y_{tt}, \quad \frac{\partial^2 Y}{\partial S \partial t} = Y_{St}, \quad Y\left(S_k^n, \frac{k}{n}\right) = Y_k.$$

Then by Taylor's expansion,

$$\begin{aligned}f_+(1) &= f_+(0) + f'_+(0) + \frac{1}{2}f''_+(0) + R_k^n \\ &= Y_k + Y_S(S_{k+1}^{n+} - S_k^n) + \frac{1}{n}Y_t + \frac{1}{2}Y_{SS}(S_{k+1}^{n+} - S_k^n)^2 + \frac{1}{2n^2}Y_{tt} + Y_{St}\frac{1}{n}(S_{k+1}^{n+} - S_k^n) + R_k^n,\end{aligned}$$

where  $R_k^n = \frac{1}{2} \int_0^1 (1-s)^2 f_+^{(3)}(s) ds$ . The expression of  $f_-(1)$  is similar to  $f_+(1)$  with  $S_{k+1}^{n+}$  replaced by  $S_{k+1}^{n-}$ . By denoting the remaining terms by  $Q_k^n$ , we have

$$\begin{aligned}&\tilde{\pi}(+; S_k^n) f_+(1) + \tilde{\pi}(-; S_k^n) f_-(1) \\ &= \left(1 + \frac{\tilde{r}_k}{n}\right)^{-1} \left[ Y_k + \frac{(r_k + \lambda_k)S_k Y_S + Y_t}{n} + \left( \frac{(b_k + \lambda_k S_k)^2 + 2(b_k + \lambda_k S_k)\sigma_k \theta_k S_k}{2n^2} + \frac{\sigma_k^2 S_k^2}{2n} \right) Y_{SS} \right. \\ &\quad \left. + \frac{1}{2n^2} Y_{tt} + \frac{(r_k + \lambda_k)S_k}{n^2} Y_{St} \right] - \gamma_k^n.\end{aligned}$$

By Lemma 2.1, the above equation equals to

$$\begin{aligned}&\left(1 + \frac{\tilde{r}_k}{n}\right)^{-1} \left[ \left(1 + \frac{r_k + \lambda_k}{n}\right) Y_k + \left( \frac{(b_k + \lambda_k S_k)^2 + 2(b_k + \lambda_k S_k)\sigma_k \theta_k S_k}{2n^2} + \frac{Y_{tt}}{2n^2} + \frac{(r_k + \lambda_k)S_k^n}{n^2} Y_{St} \right) \right] - \gamma_k^n \\ &= Y_k - \left(1 + \frac{\tilde{r}_k}{n}\right)^{-1} \frac{1}{n^2} m\left(S_k^n, \frac{k}{n}\right) - \gamma_k^n,\end{aligned}$$

where  $\gamma_k^n = -\tilde{\pi}(+; S_k^n)R_k^n - \tilde{\pi}(-; S_k^n)Q_k^n$ , and

$$m\left(S_k^n, \frac{k}{n}\right) = \left(\frac{(b_k + \lambda_k S_k)^2 + 2(b_k + \lambda_k S_k)\sigma_k \theta_k S_k}{2n^2} + \frac{1}{2n^2}Y_{tt} + \frac{(r_k + \lambda_k)S_k^n}{n^2}Y_{St}\right) + Y_k\left(\frac{\lambda_k^2 + 2r_k\lambda_k}{2} + o\left(\frac{1}{n}\right)\right).$$

Hence we obtain the following recurrent equation for  $e_k^n$ ,

$$\begin{aligned} e_k^n &= \tilde{\pi}(+; S_k^n)Y\left(S_{k+1}^{n+}, \frac{k+1}{n}\right) + \tilde{\pi}(-; S_k^n)Y\left(S_{k+1}^{n-}, \frac{k+1}{n}\right) + \left(1 + \frac{\tilde{r}_k}{n}\right)^{-1} \frac{1}{n^2}m\left(S_k^n, \frac{k}{n}\right) \\ &\quad + \gamma_k^n - \tilde{\pi}(+; S_k^n)Y_n\left(S_{k+1}^{n+}, \frac{k+1}{n}\right) - \tilde{\pi}(-; S_k^n)Y_n\left(S_{k+1}^{n-}, \frac{k+1}{n}\right) \\ &= \tilde{\pi}(+; S_k^n)e_{k+1}^{n+} + \tilde{\pi}(-; S_k^n)e_{k+1}^{n-} + \left(1 + \frac{\tilde{r}_k}{n}\right)^{-1} \frac{1}{n^2}m\left(S_k^n, \frac{k}{n}\right) + \gamma_k^n. \end{aligned}$$

By the definition of  $\tilde{\pi}(\cdot; S_k^n)$ , we obtain

$$e_k^n = \mathbb{E}_{Q_n}\left[e_{k+1}^{n+}\left(1 + \frac{\tilde{r}_k}{n}\right)^{-1} \mid \mathcal{F}_k^n\right] + \left(1 + \frac{\tilde{r}_k}{n}\right)^{-1} \frac{1}{n^2}m\left(S_k^n, \frac{k}{n}\right) + \gamma_k^n.$$

Since  $e_n^n = Y(S_n^n, 1) - Y_n(S_n^n, 1) = g(S_n^n) - g(S_n^n) = 0$ , we get

$$e_k^n = \mathbb{E}_{Q_n}\left[\sum_{i=k}^{n-1} \frac{1}{n^2}m\left(S_i^n, \frac{i}{n}\right) \frac{\beta_k^n}{\beta_{i+1}^n} + \gamma_i^n \frac{\beta_k^n}{\beta_i^n} \mid \mathcal{F}_k^n\right].$$

By the assumption that  $Y$  and its derivative satisfy linear increasing condition, there exists constants  $C_1 > 0$  and  $q$ , such that  $\left|m\left(S_i^n, \frac{i}{n}\right)\right| \leq C_1(1 + |S_i^n|^{2q})$ .

By Lemma 4.3, for  $k \leq i \leq n$  there exists constant  $C > 0$  large enough such that  $E_{Q_n}[|S_i^n|^{2q} \mid \mathcal{F}_k^n] \leq C(1 + |S_0|^{2q})$ . Therefore,

$$\begin{aligned} \mathbb{E}_{Q_n}\left[\sum_{i=k}^{n-1} \left|\frac{1}{n^2}m\left(S_i^n, \frac{i}{n}\right) \frac{\beta_k^n}{\beta_{i+1}^n}\right| \mid \mathcal{F}_k^n\right] &\leq \frac{1}{n^2} \sum_{i=k}^{n-1} \mathbb{E}_{Q_n}\left[\left|m\left(S_i^n, \frac{i}{n}\right)\right| \mid \mathcal{F}_k^n\right] \\ &\leq \frac{1}{n^2} \sum_{i=k}^{n-1} \mathbb{E}_{Q_n}[C_1(1 + |S_i^n|^{2q}) \mid \mathcal{F}_k^n] \leq \frac{1}{n^2} \sum_{i=k}^{n-1} C_1(1 + C(1 + |S_0|^{2q})). \end{aligned}$$

For the second part we can also write out the expressions of  $f_+^{(3)}(s)$ ,  $f_-^{(3)}(s)$ , they are of order  $n^{-\frac{3}{2}}$ , by analogous argument we can choose  $q$  large enough and constant  $D > 0$  satisfying

$$\mathbb{E}_{Q_n}\left[\sum_{i=k}^{n-1} \left|\gamma_i^n \frac{\beta_k^n}{\beta_i^n}\right| \mid S_k^n\right] \leq \frac{D}{\sqrt{n}}(1 + |S_k^n|^{2q}).$$

Then we can choose  $\tilde{C}$  large enough which depends on  $q$ ,  $k$  and  $n$ , such that  $|e_k^n| \leq \frac{\tilde{C}}{\sqrt{n}}(1 + |S_0|^{2q})$ . So

$$\hat{P}\left(\sup_{0 \leq t \leq 1} |\tilde{e}_t^n| \geq \epsilon\right) \leq \hat{P}\left(\sup_{0 \leq t \leq 1} \frac{\tilde{C}(1 + |\tilde{S}_0|^{2q})}{\sqrt{n}} \geq \epsilon\right) \leq \frac{\tilde{C}\left(1 + \sup_{0 \leq t \leq 1} |\tilde{S}_0|^{2q}\right)}{\epsilon\sqrt{n}} \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore,  $\sup_{0 \leq t \leq 1} |\tilde{e}_t^n| \rightarrow 0$  as  $n \rightarrow \infty$ , which means that  $\tilde{e}_t^n$  converges to zero almost surely, that is,  $\tilde{e}_t^n \Rightarrow 0$ . Combined with (4.2), we get the conclusion.  $\square$

## 5 Conclusion

In this paper, the weak convergence of discrete-time equity derivatives pricing model with default risk is proved in a no-arbitrage framework. Our results present a mathematical foundation for derivative pricing with default risk using numerical method. It remains to study the convergence for the hedging strategy.

**Acknowledgements.** We would like to thank the anonymous referee for the useful suggestions, which are a great help to improve the manuscript. We thank Xiaobin Li for his careful reading of the manuscript and useful comments. The first author would like to thank Professors Zenghu Li and Wenming Hong for their invaluable comments and encouragements during her stay in Beijing Normal University.

## References

- [1] D. Aldous, *Stopping times and tightness*, Ann. Probab, 10 (1978), 335–340.
- [2] T. Bielecki, S. Crépey, M. Jeanblanc and M. Rutkowski, *Defaultable options in a Markovian intensity model of credit risk*, Math. Finance, 18 (2009), 493-518.
- [3] T. Bielecki and M. Rutkowski, *Credit Risk: Modeling, Valuation and Hedging*, Springer-Verlag, Berlin, Heidelberg, New York, 2002.
- [4] C. Blanchet-Scalliet and M. Jeanblanc, *Hazard rate for credit risk and hedging defaultable contingent claims*, Finance Stoch., 8 (2004), 145-159.
- [5] J. Cox and M. Rubinstein, Option pricing: A simplified approach, J. Finan. Econ., 7 (1979), 229-263.
- [6] D. Duffie and P. Protter, *From Discrete to Continuous Time Finance: Weak convergence of the Financial Gain Process*, Mathematical Finance, 2 (1992), 1-15.
- [7] D. Duffie and K. Singleton, *Modeling term structure of defaultable bonds*, Revie. Finan. Studies, 12 (1999), 687-720.
- [8] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence*, John Wiley, New York, 1986.
- [9] H. He, *Convergence from discrete to continuous time contingent claims prices*, Revie. Finan. Studies, 3 (1989), 523-546.
- [10] J. Jacod, and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, Heidelberg, New York, 1987.
- [11] R. Jarrow and S. Turnbull, *Pricing Options on Financial Securities Subject to Credit Risk*, J. Finance, 50 (1995), 53–85.
- [12] D. Lando, *On Cox Processes and Credit Risky Securities*, Revie. Deriv. Research, 2 (1998), 99-120.
- [13] J. W. Nieuwenhuis and M. H. Vellekoop, *Weak convergence of three methods, to price options on defaultable assets*, Decis. Econom. Finance, 27, (2004), 87-107.